A Note on Cayley-Hamilton Theorem for Generalized Matrix Function

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Abstract

Let $\$_n$ be the symmetric group on n letters and G be a subgroup of $\$_n$. Suppose χ is an irreducible complex character of G and d_χ^G is the generalized matrix function afforded by G and χ. In this paper we characterize for which G and χ, the Cayley-Hamilton Theorem holds.

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1 Introduction

Let $\$_n$ be the symmetric group on n letters, and G be a subgroup of $\$_n$. Suppose $c: G \to \mathbb{C}$ is a complex valued function on G and let $M_n(\mathbb{C})$ denote the set of all n by n matrices over $\mathbb{C}$. The function $d_c^G: M_n(\mathbb{C}) \to \mathbb{C}$ with definition

$$d_c^G(A) = \sum_{\sigma \in G} c(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}$$

where $A = (a_{ij})$, $1 \leq i, j \leq n$, is called the *Schur function* afforded by G and c, in particular if $c = \chi: G \to \mathbb{C}$, is an irreducible complex character of G, then d_χ^G is called the *generalized matrix function* (see [3]). This function is introduced by I. Schur in [4].

Note that if $c = \epsilon: \$_n \to \mathbb{C}$ is the alternating character of $\$_n$, then $d_\epsilon^{\$_n}$ is the determinant function, i.e., $d_\epsilon^{\$_n} = \det$. If $c = 1: \$_n \to \mathbb{C}$ is the identity character 1, then $d_1^{\$_n}$ is the permanent function, i.e., $d_1^{\$_n} = \mathrm{per}$. In particular for an irreducible complex character χ of $\$_n$, $d_\chi^{\$_n}$ is called the *immanant function.* For more information about the complex irreducible characters of the symmetric group we refer the reader to [1].

Many familiar properties which hold for the determinant do not hold for d_c^G. For example as stated in [5] if A and R are n by n matrices with R invertible, then $d_c^G(A) = d_c^G(R^{-1}AR)$ does not necessarily hold. But it is proved in [2] that if R is a permutation matrix, then $d_c^G(A) = d_c^G(R^{-1}AR)$.

Definition 1 Suppose $G \leq \$_n$ and $c: G \to \mathbb{C}$ is a complex valued function on G and $A \in M_n(\mathbb{C})$. The *characteristic polynomial of A* afforded by the Schur function d_c^G is denoted by $f_A^{G,c}(x)$ and is defined by $f_A^{G,c}(x) = d_c^G(xI_n - A)$.

In the case of $G = \$_n$ and χ the irreducible complex character of G, the polynomial $f_A^{G,\chi}(x)$ is called the *χ-th immanantal polynomial of A* (see [3]). This polynomial has been studied in variety of aspects. We see from the definition, that the characteristic polynomial of A afforded by d_c^G, $f_A^{G,c}(x)$, is a polynomial over $\mathbb{C}$ with degree n, i.e., $f_A^{G,c}(x) \in \mathbb{C}[x]$. Note that if $G = \$_n$ and $c = \epsilon$ the alternating character of $\$_n$, then $f_A^{\$_n,\epsilon}(x)$ is the ordinary characteristic polynomial of A which is defined in Linear Algebra.

A property of the determinant is that the classical Cayley-Hamilton Theorem holds for it. The following example shows that the Cayley-Hamilton Theorem does not hold for d_c^G in general.

Example 2 Let $G = \{(1), (12)\} < \$_3$ and $c = \epsilon$, the alternating character of G, and

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{C}).$$

Then the characteristic polynomial of A afforded by d_c^G is

$$f_A^{G,c}(x) = d_c^G(xI_3 - A) = d_c^G \begin{pmatrix} x-1 & -1 & -1 \\ -2 & x-1 & -1 \\ 0 & 0 & x-1 \end{pmatrix} = x^3 - 3x^2 + x + 1,$$

and note that A does not satisfy $f_A^{G,c}(x)$.

In this paper we prove that if d_c^G has the property of the Cayley-Hamilton Theorem, then G must be the full symmetric group and c must be a multiple of the alternating character of G, in particular if the generalized matrix function d_χ^G has the property of the Cayley-Hamilton Theorem, then G must be the full symmetric group and χ must be the alternating character of G, i.e., the Cayley-Hamilton Theorem holds only for the determinant function, in the family of generalized matrix functions. Roughly speaking the Cayley-Hamilton Theorem is true only for determinants!

2 Main Theorem

Definition 3 Suppose $G \leq \$_n$ and $c: G \to \mathbb{C}$ is a complex valued function on G. We say d_c^G has the *Cayley-Hamilton property* if for every $A \in M_n(\mathbb{C})$ we have $f_A^{G,c}(A) = 0$.

Note that if $G = \$_n$ and $\chi = \epsilon$ the alternating character of $\$_n$, then by the classical Cayley-Hamilton Theorem $d_\epsilon^{\$_n} = \det$ has the Cayley-Hamilton property.

Lemma 4 *Let $c: \ \$_n \to \mathbb{C}$ be a complex valued function on $\$_n$. If $d_c^{\$_n}$ has the Cayley-Hamilton property, then for any singular matrix $A \in M_n(\mathbb{C})$, we have $d_c^{\$_n}(A) = 0$.*

Proof. Suppose $f_A^{\$_n,c}(x) \in \mathbb{C}[x]$ is of the form

$$f_A^{\$_n,c}(x) = a_n x^n + \cdots + a_1 x + a_0.$$

Suppose $d_c^{\$_n}(A) \neq 0$. Since $f_A^{\$_n,c}(0) = a_0 = d_c^{\$_n}(-A) = (-1)^n d_c^{\$_n}(A)$, so we obtain $a_0 \neq 0$. By hypotheses $d_c^{\$_n}$ has the Cayley-Hamilton property and therefore $f_A^{\$_n,c}(A) = 0$, i.e., $a_n A^n + \cdots + a_1 A + a_0 I = 0$ and so $A.\frac{-1}{a_0}(a_n A^{n-1} + \cdots + a_1 I) = I$, and hence A is invertible which is a contradiction. $\square$

Let $c: \ \$_n \to \mathbb{C}$ be a complex valued function on $\$_n$. If $X = (x_{ij})$ is an n by n matrix whose entries are n^2 indeterminates x_{ij} over $\mathbb{C}$, then $d_c^{\$_n}(X)$ and in particular $\det(X)$, is a polynomial over n^2 indeterminates $x_{11}, x_{12}, \ldots, x_{nn}$. In this way, we have the following Lemma. Note that this Lemma is stated in [3] (Lemma 8.3, page 267) without proof and here we provide a proof for it.

Lemma 5 $\det(X)$ *is an irreducible polynomial over* $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$.

Proof. If $\det(X)$ is reducible, then we have $\det = P.Q$, where P and Q are nonconstant polynomials in $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$. If x_{ij} appears in P for some i and j, then since det is linear in each variable the polynomial Q cannot contain terms involving x_{ik} or x_{kj}, $1 \leq k \leq n$, which are in the same column or row as x_{ij}. So Q is constant, and this is a contradiction. $\square$

Lemma 6 *Let $c: \ \$_n \to \mathbb{C}$ be a complex valued function on $\$_n$. If $d_c^{\$_n}$ has the Cayley-Hamilton property, then $\det(X) \mid d_c^{\$_n}(X)$ in $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$, where $X = (x_{ij})$ is an arbitrary n by n matrix over $\mathbb{C}$.*

Proof. Consider $I_1 = \langle d_c^{\$_n}(X) \rangle$ and $I_2 = \langle \det(X) \rangle$, the ideals generated by $d_c^{\$_n}(X)$ and $\det(X)$ respectively in $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$. Suppose for the ideal I of $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$, $V(I)$ denotes the affine $\mathbb{C}$-variety in $\mathbb{C}^{n^2}$,

$$V(I) = \{(a_{11}, a_{12}, \ldots, a_{nn}) \in \mathbb{C}^{n^2} \mid f(a_{11}, a_{12}, \ldots, a_{nn}) = 0, \text{ for all } f \in I\},$$

and for the subset Y of $\mathbb{C}^{n^2}$, $J(Y)$ denotes the set

$$J(Y) = \{f \in \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}] \mid f(a_{11}, a_{12}, \ldots, a_{nn}) = 0, \text{ for all } (a_{11}, a_{12}, \ldots, a_{nn}) \in Y\}.$$

By Lemma 4, we have $V(I_2) \subseteq V(I_1)$ and therefore $J(V(I_1)) \subseteq J(V(I_2))$ so by Hilbert-Nullstellensatz, we obtain $\sqrt{I_1} \subseteq \sqrt{I_2}$. Since $I_1 \subseteq \sqrt{I_1}$, so $I_1 \subseteq \sqrt{I_2}$. Because $d_c^{\$_n}(X) \in I_1$, we have $d_c^{\$_n}(X) \in \sqrt{I_2}$ and this means there exists $k \in \mathbb{N}$ such that $\left(d_c^{\$_n}(X)\right)^k \in I_2$ and so $\det(X) \mid \left(d_c^{\$_n}(X)\right)^k$. But by Lemma 5, $\det(X)$ is irreducible over $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$ and therefore $\det(X) | d_c^{\$_n}(X)$ in $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$. $\square$

Corollary 7 *Let $c : \$_n \to \mathbb{C}$ be a complex valued function on $\$_n$. If $d_c^{\$_n}$ has the Cayley-Hamilton property, then there is a constant $k \in \mathbb{C}$, such that $c = k\epsilon$, where ϵ is the alternating character of $\$_n$.*

Proof. By Lemma 6, $\det(X)|d_c^{\$_n}(X)$ in $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$, but $\det(X)$ and $d_c^{\$_n}(X)$ are of degree one in each indeterminate, therefore $d_c^{\$_n}(X) = k \det(X)$, where $k \in \mathbb{C}$ is a constant, and so $c = k\epsilon$. $\square$

Corollary 8 *Let $G \lneqq \$_n$ and $c : G \to \mathbb{C}$ be a nonzero complex valued function on G. Then d_c^G does not have the Cayley-Hamilton property.*

Proof. Assume d_c^G has the Cayley-Hamilton property. Define the function $c' : \$_n \to \mathbb{C}$ by

$$c'(\sigma) = \begin{cases} c(\sigma) & , \text{ if } \sigma \in G, \\ 0 & , \text{ if } \sigma \in \$_n \setminus G. \end{cases}$$

Then for all $A = (a_{ij}) \in M_n(\mathbb{C})$,

$$d_c^G(A) = \sum_{\sigma \in G} c(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)} = \sum_{\sigma \in \$_n} c'(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)} = d_{c'}^{\$_n}(A),$$

therefore $d_c^G = d_{c'}^{\$_n}$. Because d_c^G has the Cayley-Hamilton property, $d_{c'}^{\$_n}$ satisfies this property too and hence by Corollary 7, $c' = k\epsilon$, where $k \in \mathbb{C}$ is a constant. By definition of c' we obtain $k = 0$, so $c' = 0$ and therefore $c = 0$. This contradiction shows d_c^G does not have Cayley-Hamilton property. $\square$

Main Theorem *Let $G \leq \$_n$ and χ be an irreducible complex character of G. Then d_χ^G has the Cayley-Hamilton property if and only if $G = \$_n$ and $\chi = \epsilon$, the alternating character of $\$_n$, i.e., $d_\chi^G = \det$.*

Proof. If d_χ^G has the Cayley-Hamilton property, then by Corollary 8, $G = \$_n$. So $d_\chi^G = d_\chi^{\$_n}$ and by Corollary 7, we have $\chi = k\epsilon$, where $k \in \mathbb{C}$ is a constant. But χ is an irreducible character of $G = \$_n$ and therefore we have $k = 1$, so $\chi = \epsilon$, hence $d_\chi^G = d_\epsilon^{\$_n} = \det$. The converse of the Theorem holds by the classical Cayley-Hamilton Theorem. $\square$

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